Noise in Resistances and Electron Streams

By J. R. PIERCE

TECHNICALLY correct results in a field are achieved initially in diverse and often confusing and complicated ways. Sometimes, such results are later brought together to give them a more unified form and a sounder basis; such critical summary and exposition is of great value. In quite another way, a worker who uses results established in a field will discover many plausible reasons for believing the results, and he will find eventually that an air of inevitability and "understanding" pervades the subject. Such "understanding" is not to be confused with the process of rigorous proof carried out step by step, but it can help in organizing and making use of a body of related material.

The field of "noise", especially as it affects electron devices and communications in general, is one particularly troublesome to engineers. The sound work on the subject has commonly involved mathematics and especially statistical ideas unfamiliar to many who must deal with the practical problems of noise. In early papers on noise, a great deal of heat was generated in acrimonious controversy between two schools, one of which assigned a uniform noise spectrum to certain noise sources, while the other held this to be inadmissible and got identical answers by more recondite means. Happily, a recent paper by S. O. Rice[1] clearly presents both approaches. Rice's paper further provides a fine broad summary of noise problems together with considerable original material. It does not extend far into the field of electronics.[2]

The reader who has sufficient time could achieve a profound "understanding" of the circuit aspects of noise by reading Rice's paper. The understanding would involve familiarity with much mathematics useful in itself. To many engineers, however, this might prove a lengthy and painful process.

The writer proposes to present here a series of plausible arguments for believing certain facts about noise. Both simple circuit considerations and "electronic" effects (as, space charge reduction of noise) are included. The arguments presented are not intended to be original and it is not claimed that they are rigorous; they do seem to be easily understood, and to help in remembering and in using some important practical material. Starting points of the arguments, or "postulates", have been chosen on the basis of familiarity, not simplicity. No effort is made to point out all of the hidden assumptions in the arguments, but a few important ones are indicated.

An initial warning should be made that quantum effects treated in Nyquist's original paper on Johnson noise, but afterwards much neglected, are entirely disregarded here.

I. Johnson Noise[3]

In 1926, in an investigation of amplifiers with exceedingly high grid resistances, J. B. Johnson discovered that a resistance acts as a noise generator having an open-circuit voltage with a mean square value

$$\overline{v^2} = 4kTRB. \tag{1}$$

Here and subsequently, lower case letters v and i will be used in referring to noise voltages and currents. In (1), $\overline{v^2}$ is the mean square value of noise voltage components of frequency lying in a small bandwidth B (sometimes

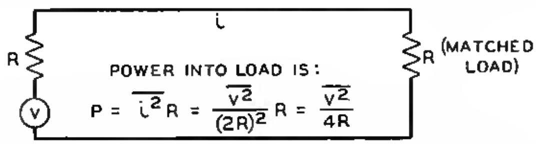

Fig. 1—Relations between noise power, noise voltage and noise current can be derived by assuming the noise source to be a voltage in series with a resistance.

called df or Δf), k is Boltzman's constant, and R is resistance. We easily see from Fig. 1 that the maximum noise power which can be made to flow from the resistance into a load (that which will flow into a matched load) is

$$P = \frac{\overline{v^2}}{4R} = kTB. \tag{2}$$

This "available noise power" is a convenient alternative formulation.

If an impedance has a reactive as well as a resistive component, the open circuit noise is given by (1) where R is the resistive component; if an admittance has a conductance G the noise may be represented as an impressed current (that which flows when the admittance is short circuited) of magnitude

$$\overline{i^2} = 4kTGB. \tag{3}$$

We see from (1) that if two resistances are connected in series, the total *squared* noise voltage is the sum of the *squares* of the noise voltages produced by the resistances separately, and from (3) we see that the noise currents of conductances connected in shunt also add by summing squares. This rule of addition holds for adding the noise of all independent sources. Of course, if noise from the same noise source reaches a point by different paths, the

voltage or current components near any frequency should be added directly with due regard for phase.

Johnson noise is related to many physically similar phenomena such as Brownian motion and the random fluctuations in position observed in the coils of very sensitive galvanometers.

The simplest derivation of (1), (2) or (3) is that given by Nyquist[4] in a companion paper to Johnson's. Consider a long lossless transmission line of length L terminated at each end in resistances equal to its characteristic impedance. Imagine line and terminations in thermal equilibrium at a temperature T, as shown in Fig. 2. If electrical energy flows from the resistance at 1 to that at 2, then equal energy must then flow from 2 to 1, as any net gain or loss of energy would violate the second law of thermodynamics.

Now, suppose that we suddenly close the switches at 1 and 2, short circuiting the ends of the line. The line now becomes a resonator, having resonant

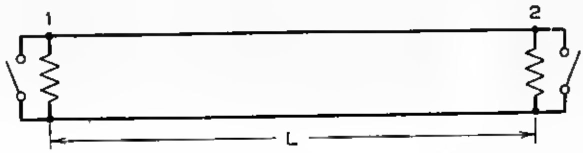

Fig. 2—Two resistances terminating a transmission line act as generators of thermal noise power traveling along the line.

frequencies such that the line is n half wavelengths long. The resonant frequencies will be

$$f = n(c/2L). \tag{4}$$

Here n is an integer and c is the velocity of light. The frequencies are separated by frequency intervals

$$\Delta f = (c/2L). \tag{5}$$

The energy which originally flowed right to left and left to right between the resistances is now reflected at the ends. It may be expressed as the thermal energy associated with the resonant modes of the line. According to statistical mechanics, there is an energy kT associated with each resonant mode. The energy per unit bandwidth is obtained by dividing this by the frequency interval between modes, given by (5) and is

$$w = kT/\Delta f = kT/(c/2L). \tag{6}$$

Since it takes a wave a time L/c to pass completely through the line, this energy w represents the energy per unit bandwidth which flowed into the

line from both resistances over a period L/c. If p is the power per unit bandwidth from one resistance, then

$$\begin{aligned} 2p(L/c) &= w = kT/(c/2L) \\ p &= kT. \end{aligned} \tag{7}$$

Or, we may say that the power flow from a resistance into a matched load (the available power) is, for a bandwidth B

$$P = kTB. \tag{8}$$

Sometimes it may be desired to know the mean squared fluctuation voltage integrated over all frequencies. Carrying out such an integration for the voltage between a pair of terminals connected by a complicated network would seem to be a difficult procedure. However, if the pair of terminals is shunted by a capacitance, the integrated fluctuation voltage can be obtained by direct application of the principles of statistical mechanics.

In a lumped network composed of capacitive, inductive and resistive elements* each capacitance and each inductance constitutes a degree of freedom; that is, the electrical state of the network can be specified completely by specifying the voltage across each capacitance and the current in each inductance**. According to statistical mechanics, the average stored energy per degree of freedom is $kT/2$. The stored energy in a capacitance is $Cv^2/2$. Thus, the mean squared noise voltage of all frequencies across a capacitance C must be

$$\overline{v^2} = kT/C. \tag{9}$$

Similarly, the mean squared noise current of all frequencies flowing in an inductance L is

$$\overline{i^2} = kT/L. \tag{10}$$

We have conveniently thought of Johnson noise as generated in the resistances in a network. We need not change this concept and say that the voltage and current of (9) and (10) are generated in the capacitance or inductance any more than we would say that the thermal velocities of molecules are generated by the molecules' mass. Relations (9) and (10) merely represent necessary consequences of the laws of statistical mechanics as, indeed, does (1).

It is of some interest to illustrate the use of (9) and its connection with (1)

* Strictly, such a lumped network is an unrealizable ideal. There are no pure capacitances, inductances, or resistances. The conditions under which actual condensers, coils and resistors can be represented satisfactorily by these idealizations must be judged by measurement or calculation or by past experience or intuition.

** In enumerating the degrees of freedom, capacitances in series or shunt are lumped together as one element; the same holds true for inductances.

hy a very simple example. Consider a resonant circuit consisting of a capacitance C, an inductance L and a resistance R_0, all in parallel. The resistive component of the impedance across this circuit is

$$R = \frac{R_0}{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2} \tag{11}$$

$$Q = R_0\omega_0 C = R_0/\omega_0 L \tag{12}$$

$$\omega_0 = 1/\sqrt{LC}. \tag{13}$$

Here ω_0 is the resonant frequency of the circuit and Q has its usual meaning.

From (1) we see that as R_0, the resistance at resonance ($\omega = \omega_0$) is made higher, the noise voltage for frequencies near resonance increases. However, if we regard ω_0 and C in (12) as fixed, we see that as R_0 is increased the Q of the circuit is increased, the frequency range over which R is high is decreased, and R actually becomes lower far from resonance. (9) tells us that the mean square noise voltage integrated over all frequencies remains constant as R_0 is changed.

It is found that for a high Q circuit, the noise is much like a carrier of frequency ω_0 modulated hy low-frequency noise. If we let the radian frequency of this "noise modulation" he $(\omega - \omega_0)$, then the mean square amplitude of the noise modulation varies with frequency ahout as R given hy (11) varies with $(\omega - \omega_0)$.

II. Schottky Noise or Shot Noise

In 1918 Schottky[6] descrihed the "Schrot-Effekt": the noise in vacuum tuhes due to the corpuscular nature of the electron convection current. This is commonly known as "shot noise." The magnitude of this noise is usually derived by means quite different from those used here.

Johnson noise is necessarily associated with any electrical resistance, whatever its nature. Now, consider a close spaced planar diode shown in Fig. 3 consisting of two opposed emitting cathodes, each emitting a current I_0. Suppose the whole diode is held at the same temperature. There are no hatteries or other sources of power aside from thermal energy; the only electrical energy flow must then he Johnson noise, ascrihahle to the resistance of the diode.

Assume that the cathodes hoth have the same uniform work function. Then when the diode is short circuited, each electron emitted from cathode 1 will reach cathode 2, and each electron emitted from cathode 2 will reach cathode 1.* If cathode 2 were made negative, all the electrons from 2 would

* It is here assumed that I_0 is small enough so that depression of potential due to space charge is avoided.

continue to reach 1, but some of the low-velocity electrons leaving 1 would be turned back from 2.

It is well known[6] that if a Maxwellian velocity distribution is assumed for the electrons leaving 1, the electrons which can overcome the retarding field and reach 2 are found to constitute a current

$$I = I_0 \epsilon^{eV/kT}. \tag{14}$$

Here I_0 is the total current carried by electrons leaving 1 and V is the voltage of 2 with respect to 1, which has been assumed to be negative.

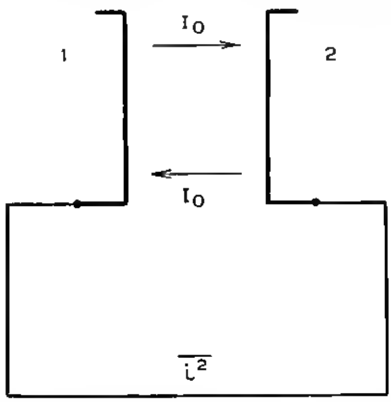

Fig. 3—An electronic resistance formed by two opposed cathodes at the same temperature acts as a generator of tbermal noise.

By differentiating (14) we can obtain the diode conductance G at $V = 0$, and we find

$$G = \frac{e}{kT} I_0. \tag{15}$$

From (3) when the diode is short circuited and the voltage is zero we have a mean square noise current

$$\overline{i^2} = 4kTGB = \frac{eI_0}{kT}(4kTB) = 4eI_0 B. \tag{16}$$

This noise is the sum of the noise due to two independent noise sources (the noise in the two currents I_0). That due to either current I_0 is*

$$\overline{i^2} = 2eI_0 B. \tag{17}$$

* In this section, we are concerned with short transit angles only and no distinction need be made between the current induced in the circuit, i, and the electron convection current.

This is the expression for shot noise in a randomly emitted current, as in temperature limited emission or in photo electric emission.

III. Noise Other Than Shot Noise: Electron Multipliers

Let us consider a class of systems in which the average output current is proportional to the average input current, in which an electron of charge, e entering produces an output charge, ne instantaneously, and in which the probability that any electron will produce n electrons is p_n.

If the input current is I_0, the average output current is

$$I = \bar{n} I_0 \tag{18}$$

$$\bar{n} = \sum_n n p_n . \tag{19}$$

It is easy to persuade ourselves that any frequency component of current, noise or signal, will produce an output current $\bar{n}$ times as great; this happens to be true, and we will use the fact.

Let us consider our device when it has randomly emitted electrons as an input. At the output we will see appear groups of 1, 2, 3 etc. electrons, each group caused by the entrance of a single electron. If I_0 is the total input current, the output current consisting of groups of n electrons is

$$I_n = n I_0 p_n . \tag{20}$$

Each group carries a charge ne. We may now use (17) to write the noise in the part of the current carried by groups of n electrons, replacing the electronic charge, e, by the group charge, ne

$$\overline{i_n^2} = 2(ne)(n I_0 p_n) B. \tag{21}$$

As there is no correlation between entering electrons, the total mean square output noise current will be the sum of the noise components carried by groups consisting of different numbers n of electrons. Summing (19) with respect to n we obtain

$$\overline{i_t^2} = 2eI_0 B\overline{n^2} \tag{22}$$

$$\overline{n^2} = \sum_n n^2 p_n . \tag{23}$$

Now, the input has been taken as having shot noise. A part of the noise output is to be attributed to this input shot noise amplified by the device; that is, it will be $\bar{n}^2$ times the input shot noise.

$$\overline{i_s^2} = \bar{n}^2 2eI_0 B. \tag{24}$$

The part of the noise output current due to the fact that an electron does

not produce $\bar{n}$ electrons, but may produce 0, 1, 2 ··· etc. electrons, must be the difference between (22) and (24), or

$$\overline{i_1^2} = 2eI_0 B(\overline{n^2} - \bar{n}^2). \tag{25}$$

The quantity in parentheses is the mean square deviation in n.*

If the input current has any noise components $\overline{i_n^2}$, then the total noise output component will be

$$\overline{i^2} = \overline{i_1^2} + \bar{n}^2\,\overline{i_n^2}. \tag{26}$$

By applying (26) successively to stage after stage the noise output of a multistage electron multiplier can be evaluated (if one knows $(\overline{n^2} - \bar{n}^2)$).[7]

Wonder is sometimes expressed that current can be noisier than shot noise, in which the time of electron arrival is purely random. Obviously, we can have more than shot noise only if there is something non-random about the time of electron arrival, and the argument above discloses just what this is; it is the arrival of electrons in bunches.** We can easily see how erratic even large currents would be if electrons were bound together in groups having a total group charge of a coulomb, all the electrons in a group arriving simultaneously. Reverting to our shot noise formulas, we may illustrate this by assuming a perfect multiplier with a shot noise input, in which each input electron produces exactly N output electrons. Arguing from the shot noise equation (17) and replacing e by Ne we should expect an output noise current

$$\overline{i^2} = 2(Ne)I_1 B \tag{27}$$

where I_1 is the output current; we get exactly the same result by assuming the input noise current squared amplified by N^2

$$\begin{aligned}\overline{i^2} &= (2eI_0B)N^2\\ &= 2(Ne)\,(NI_0)B\\ &= 2(Ne)\,I_1B\,.\end{aligned} \tag{28}$$

* The mean square deviation is the sum with respect to n of the square of the deviation from the mean value of n, $\bar{n}$.

$$\Sigma\,(n - \bar{n})^2 p_n = \Sigma\, n^2 p_n - 2\bar{n}\,\Sigma\, np_n + \bar{n}^2\,\Sigma\, p_n.$$

The summation in the first term is $\overline{n^2}$, that in the second term is $\bar{n}$ and that in the third term is unity. Hence

$$\Sigma\,(n - \bar{n})^2 p_n = (\overline{n^2} - \bar{n}^2).$$

** Anything, (such as transit time difference for electrons within a bunch) which tends to break up the bunches will reduce the noise—and the signal as well. Such noise reduction involves a return to a more nearly random flow.

Conversely, we are led to wonder whether a current less bunched than that produced by random emission might not have less noise. The most smoothly distributed current we can imagine is that of f_0 electrons per second emitted at evenly spaced intervals. Obviously, such a current will have a spectrum consisting of frequencies nf_0, integral multiples of f_0. Thus for $f < f_0$, there will be no "noise" and similarly for $f_0 < f < 2f_0$, $2f_0 < f < 3f_0$, etc.

For a current of 10 ma, $f_0 = 6.3 \times 10^{16}$; thus, even for small currents an evenly spaced emission would have no a-c components in the radio-frequency range; this is a comforting thought in considering space-charge reduction of noise, which is discussed in section 5. However, purely to satisfy our curiosity we may pursue the matter a little further. If we assume that each electron constitutes an instantaneous pulse of current, a simple harmonic analysis shows that the a-c current component of frequency $n f_0$ will have a mean square value

$$\overline{i_n^2} = 2eI_0 f_0. \tag{29}$$

Thus, in each interval f_0 wide centered about a frequency $n f_0$ there will be a mean squared a-c current equal to that which would be associated with the same band for random emission with the same current. By making the emission regular we have not reduced the mean square "noise" current in a broad frequency range; we have merely changed its frequency distribution from a uniform distribution to a distribution of sharp, high peaks.

IV. Partition Noise

Consider a tetrode, shown in Fig. 4, with a cathode current I_c, a screen current I_s, and a plate current I_p.

The grid current is taken as zero. Suppose that the screen is very fine, so that every electron leaving the cathode has the same chance of striking the screen, regardless of its point of departure. We may now regard the function of the screen as that of a peculiarly simple electron multiplier, for which n can be zero (electron striking screen) or 1 (electron passing screen).

The probability of an electron passing the screen is I_p/I_c. Accordingly, from (19) and (23),

$$\bar{n} = I_p/I_c \tag{30}$$

$$\overline{n^2} = I_p/I_c. \tag{31}$$

Suppose we write the noise in the cathode current as

$$\overline{i^2} = \Gamma^2 2eI_c B \tag{32}$$

Here Γ^2, a factor less than unity, is introduced to account for the "space charge noise reduction" in space charge limited flow.

Now, by applying (25) and (26) we obtain for the noise in the plate current

$$\begin{aligned} \overline{i_p^2} &= 2eI_cB(I_p/I_c - (I_p/I_c)^2) + \Gamma^2 2eI_cB(I_p/I_c)^2 \\ \overline{i_p^2} &= 2eI_pB(1 - (1 - \Gamma^2)(I_p/I_c)). \end{aligned} \tag{33}$$

It is to be noted that if $\Gamma^2 = 1$, that is, if the cathode current is random, the noise in the plate current is purely shot noise. The screen cannot make the plate current noisier than shot noise since it does not act to produce bunches of electrons.

The noise in the screen current can be obtained by substituting I_s for I_p

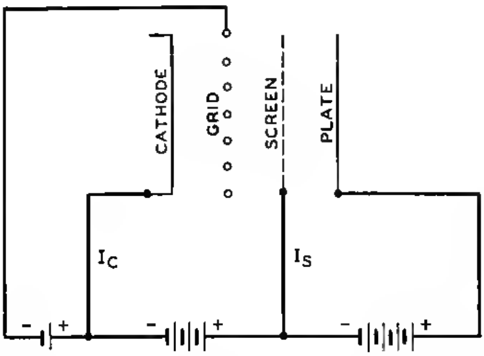

Fig. 4—Electrons randomly hitting or missing the screen grid make a tetrode noisier than a triode.

in (33). There is a correlation between the screen and plate noise currents; the total noise in the screen current plus the plate current must, of course, be

$$\overline{i_s + i_p^2} = \overline{i_c^2} = \Gamma^2 2eI_0B \tag{34}$$

and not the sum of $\overline{i_p^2}$ and $\overline{i_s^2}$.

Partition noise has been discussed by Thompson, North and Harris.[8]

V. Space Charge Reduction of Noise

In this section an approximate derivation of noise in a space charge limited diode will be presented. The derivation leads to an expression valid for many practical tubes and illustrates the nature of the noise in space charge limited flow.

Consider a parallel plane diode of unit area and spacing x_1 with an applied voltage V_0, as shown in Fig. 5. When the voltage is applied, the electron convection current in the diode rises to value I_0. Neglecting thermal velocities of electron emission, this current is such that the electronic

"space charge" associated with it causes the voltage gradient at the cathode surface to be zero. A greater current would mean a negative gradient at the cathode and hence no emission; a smaller current would mean a positive gradient at the cathode and unlimited emission. On this basis Child's law is derived, which gives the current per unit area I_0 in amperes in terms of the voltage V_0 and the spacing in centimeters x as

$$I_0 = (2.33)\, 10^{-6} V_0^{3/2}/x^2 . \tag{35}$$

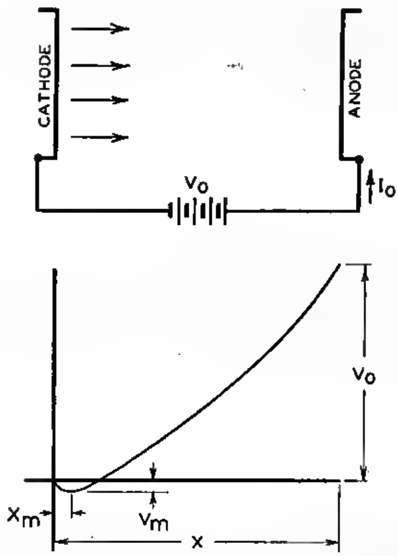

Fig. 5—Part of the electrons leaving the cathode of a diode are turned back before reaching the potential minimum; others proceed to the anode. Ordinarily the greater amount of noise is associated with the space between the potential minimum and the anode.

From (35) we can obtain a useful relation for the conductance G

$$G = \partial I_0/\partial V_0 = (3/2)(I_0/V_0). \tag{36}$$

The resistance R is

$$R = 1/G = (2/3)(V_0/I_0). \tag{37}$$

In actual diodes, the electrons are emitted from the cathode with a thermal velocity distribution; a potential minimum of some negative voltage V_m is formed at some distance x_m from the cathode surface. If the magnitude of the emitted electron current is I_e and the actual current passing the potential minimum is I_0, then because of the Maxwellian velocity distribution we have

$$\begin{aligned} I_0 &= I_e \epsilon^{eV_m/kT} \\ &= I_e \epsilon^{11,600 V_m/T}. \end{aligned} \tag{38}$$

Ordinarily, the magnitude of V_m is very small compared with V_0 ; I_0 is very small compared with I_e and x_m is very small compared with x.

Suppose V_m were held constant, say, by putting a conducting plane of potential V_m at x_m. Then, the electrons which pass this plane are quite independent of the low energy electrons which are turned back, and hence in the current passing x_m there will be pure shot noise.

$$\overline{i^2} = 2eI_0B. \tag{39}$$

Now suppose we change V_m. The change in I_0 will be, from (38),

$$dI_0 = dV_m/R_m \tag{40}$$

$$R_m = (eI_0/kT)^{-1}. \tag{41}$$

If we use a constant current instead of a constant voltage d-c supply, then V_m must fluctuate in such a way as to cause a current equal and opposite to (39), or, there must be a fluctuating voltage $\overline{v_m^2}$ such that

$$\begin{aligned} \overline{v_m^2} &= 2eI_0BR_m^2 \\ &= (1/2)4kTR_mB. \end{aligned} \tag{42}$$

Suppose we consider the noise fluctuation of the anode voltage of a space charge limited diode supplied from a constant-current source. If there were no fluctuations in the voltage drop between the potential minimum at x_m and the anode at x, (42) would give the noise voltage fluctuation of such an "open circuited" diode. Actually, much larger fluctuation voltages are observed, and we must conclude that they arise in the space between the potential minimum and the anode. As the current is constant in this region (by definition—we have assumed a constant-current supply) we are forced to conclude that such fluctuations are due to a variation of mean electron speed in this region. The field at x_m is necessarily zero. If, with a constant current, electrons travel more rapidly between x_m and the anode, there is less electronic charge everywhere in this region, the rate of change of field with distance, and hence, the field, are everywhere smaller, and the voltage between x_m and the anode at x will be smaller.

It is somewhat involved to treat the problem of multi-velocity flow exactly; this has been done by Rack[9] and others[8,10,11]; however, Rack has shown that an approximate treatment yields very nearly the correct result over a fairly wide range of conditions. In this approximation, the stream of electrons with many velocities and a fluctuating mean velocity is replaced by a stream in which all electrons have the same velocity, and this has a mean square fluctuation equal to that of the multi-velocity stream.

Let us now measure x from the potential minimum. Suppose we consider an electron which passed the potential minimum ($x = 0$) at $t = 0$.

The field at the potential minimum is zero. The charge which has flowed in behind the electron at the time t is $-t I_0$. Hence, from Gauss's theorem the potential gradient is

$$\partial V/\partial x = I_0 t/\epsilon \tag{43}$$

where ϵ is the dielectric constant of vacuum. We have for the acceleration

$$\ddot{x} = \frac{e}{m}\frac{I_0 t}{\epsilon}. \tag{44}$$

If at the time $t = 0$ (at the potential minimum), $x = 0$, $\dot{x} = \dot{x}_0$

$$\dot{x} = \frac{e}{m}\frac{I_0}{2\epsilon} t^2 + \dot{x}_0 \tag{45}$$

$$x = \frac{e}{m}\frac{I_0}{6\epsilon} t^3 + \dot{x}_0 t. \tag{46}$$

Now the voltage V between the potential minimum and any point x must be such that

$$\dot{x}^2 - \dot{x}_0^2 = 2\frac{e}{m} V_0 \tag{47}$$

$$V_0 = \frac{1}{2\dfrac{e}{m}}\left(\frac{e}{m}\frac{I_0}{2\epsilon}\right)^2 t^4 + \frac{I_0}{2\epsilon} t^2 \dot{x}_0. \tag{48}$$

At any fixed point x, if we vary $\dot{x}_0$ by a small amount $d\dot{x}_0$, we find by differentiating (46)

$$\frac{dt}{d\dot{x}_0} = -\frac{t}{\left(\dfrac{e}{m}\dfrac{I_0}{2\epsilon} t^2 + \dot{x}_0\right)}. \tag{49}$$

From (48)

$$dV_0 = \frac{I_0 t}{\epsilon}\left(\frac{e}{m}\frac{I_0}{2\epsilon} t^2 + \dot{x}_0\right) dt + \frac{I_0}{2\epsilon} t^2 d\dot{x}_0. \tag{50}$$

Using (49)

$$dV_0 = -\frac{I_0}{2\epsilon} t^2 d\dot{x}_0. \tag{51}$$

It now remains to evaluate t. For most cases, the thermal velocities at the potential minimum are so small compared with the velocities in most of the region between the minimum and the anode that we can take the value of t for $\dot{x}_0 = 0$. Then, from (45) and (47)

$$t^2 = \left(\frac{e}{m}\frac{I_0}{2\epsilon}\right)^{-1}\left(2\frac{e}{m} V_0\right)^{1/2}. \tag{52}$$

From (51) and (52)

$$dV_0 = -2^{1/2}\left(\frac{e}{m}\right)^{-1/2} V_0^{1/2}\, d\dot{x}_0. \tag{53}$$

Now, if $\overline{(d\dot{x}_0)^2}$ is the mean square fluctuation in velocity, the mean square fluctuation in voltage will be

$$\overline{v^2} = 2(e/m)^{-1} V_0\, \overline{(d\dot{x}_0)^2}. \tag{54}$$

The assumptions leading to (54) are those leading to Child's law, and thus we can use (37) in connection with (54), giving

$$\overline{v^2} = 3(e/m)^{-1} I_0 R\, \overline{(d\dot{x}_0)^2}. \tag{55}$$

It now remains to evaluate $\overline{(d\dot{x}_0)^2}$, the mean square fluctuation in the velocity of the electrons passing the potential minimum; to do this, we return to (25). Suppose N is the number of input electrons per second. The output current can then be written

$$I_1 = \bar{n}Ne \tag{56}$$

and we can call the fluctuation in it

$$\overline{i^2} = \overline{(\delta\bar{n}Ne)^2}. \tag{57}$$

Equation (25) applies for no fluctuation in I_0 and hence for no fluctuation in N; e is a constant, and thus we may write (25) as

$$\overline{(\delta\bar{n})^2} = \frac{2B}{N}\,(\overline{n^2} - \bar{n}^2). \tag{58}$$

We may generalize this to say that each electron has a probability p of producing some effect of magnitude n and the fluctuation in the magnitude of the effect is $\overline{(\delta\bar{n})^2}$. Before, we said that an electron had a probability p of producing n secondaries. Now we will say instead that an electron has an uncorrelated probability p of having a velocity u, and obtain for the mean fluctuation in the velocity, $\overline{(d\dot{x}_0)^2}$

$$\overline{(d\dot{x}_0)^2} = \frac{2B}{N}\,(\overline{u^2} - \bar{u}^2). \tag{59}$$

In a Maxwellian distribution, the number of electrons passing a plane perpendicular to the direction of motion per second having velocities lying in the range du at u is

$$dn = Au\epsilon^{-(mu^2/2kT_c)}\, du. \tag{60}$$

Here T_c is cathode temperature. We see $\bar{u}$ and $\overline{u^2}$ are

$$\bar{u} = \frac{\int_0^\infty u^2 \epsilon^{-(mu^2/2kT_c)}\, du}{\int_0^\infty u\epsilon^{-(mu^2/2kT_c)}\, du} = \sqrt{\frac{\pi}{2}} \sqrt{\frac{kT_c}{m}} \tag{61}$$

$$\overline{u^2} = \frac{\int_0^\infty u^3 \epsilon^{-(mu^2/2kT_c)}\, du}{\int_0^\infty u\epsilon^{-(mu^2/2kT_c)}\, du} = 2\,\frac{kT_c}{m}. \tag{62}$$

Accordingly

$$\overline{u^2} - \bar{u}^2 = \tfrac{1}{2}(4 - \pi)\,\frac{kT_c}{m} \tag{63}$$

Combining (63) with (59) we obtain

$$\overline{(d\dot{x}_0)^2} = \frac{B}{N}\,(4 - \pi)\,\frac{kT_c}{m}. \tag{64}$$

Combining (62) with (53) and remembering that $I_0 = Ne$ we find the mean square open circuit noise voltage to be

$$\begin{aligned} \overline{v^2} &= 3(4 - \pi)\, kT_cRB \\ &= (.644)\, 4kT_cRB. \end{aligned} \tag{65}$$

This is the chief contribution to noise in a space charge limited diode.

Usually R is substantially equal to the plate resistance of the diode (it does not include effects on the cathode side of the potential minimum). Hereafter R will be treated as the total plate resistance of the diode.

VI. Noise in Triodes and Pentodes

Consider the triode shown in Fig. 6. Here we have a cathode, a grid, and a plate. The input admittance of the tube is represented in the diagram by the grid-cathode capacitance C_1 and the grid-plate capacitance C_2. The resistance R_n is a fictitious noise resistance which will be evaluated later. It is assumed to act between the input admittance of the tube and the controlling action of the grid; no current can flow in R_n because the grid as indicated in the diagram is presumed to present an open circuit.

We will regard the cathode-grid region of the triode as an "equivalent diode." The anode voltage of the diode is taken as

$$V_0 = (V_g + V_p/\mu). \tag{66}$$

Here V_g is the grid voltage and V_p the plate voltage of the triode. If the plate voltage is held constant and μ is taken as constant

$$dV_0 = dV_g. \tag{67}$$

Hence, under these conditions

$$\partial I_0/\partial V_0 = G = \partial I_0/\partial V_g. \tag{68}$$

Here G is the conductance of the equivalent diode, the reciprocal of R which appears in (65), and is also the transconductance of the triode.

As we wish to calculate the noise with no a-c grid or plate voltage, and as these through (64) specify the plate voltage V_0 of the equivalent diode,

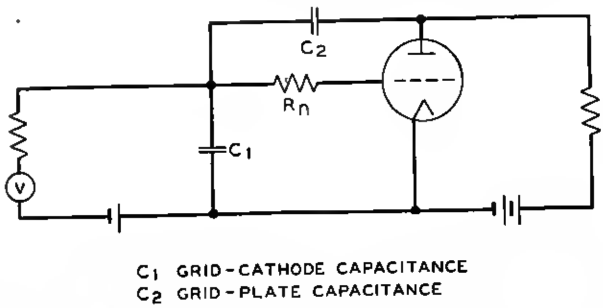

Fig. 6—Low-frequency noise in a triode can be ascribed to a fictitious noise resistance R_n, acting into an open circuit to cause voltage fluctuations on the grid.

the equivalent diode may be regarded as short-circuited. Hence, the noise current will be

$$\begin{aligned}\overline{i^2} &= \overline{v^2}/R^2 \\ &= (.644)\ 4kT_a GB.\end{aligned} \tag{69}$$

If we express this as shot noise reduced by a factor Γ^2 we obtain

$$\begin{aligned}\overline{i^2} &= 2eI_0\Gamma^2 B \\ \Gamma^2 &= (.644)\,\frac{2kT_cG}{eI_0}\,.\end{aligned} \tag{70}$$

Often, the noise expressed by (69) is ascribed as a fictitious noise resistance R_n, at room Temperature T, connected between the grid-cathode capacitance and the "controlling action" of the grid as shown in Fig. 6. This fictitious resistance looks into a complete open circuit; hence, it has a noise voltage

$$\overline{v^2} = 4kTR_nB \tag{71}$$

and produces a noise plate current (for zero load resistance)

$$\overline{i^2} = 4kTR_nBG^2. \tag{72}$$

Comparing (69) with (72) we find

$$R_n = (.644/G)\ (T_c/T_0). \tag{73}$$

Here T_0 is a reference temperature, usually taken as 290° K. The effect of load impedance on signal from this fictitious resistance is treated by purely circuit means.

In pentodes there is noise according to (69) and in addition there is partition noise according to (33). By taking Γ^2 from (70) and equating the noise current given by (33) to (72) the fictitious "noise resistance" of a pentode can be evaluated in terms of g, I_p/I_c and T_c/T.

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